

# Stabilization of coupled second order systems with delay

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## Abstract

In this paper we characterize the output feedback stabilization of some coupled systems with delay. The proof of the main result uses the method introduced in Ammari and Tucsnak [4] where the exponential stability for the closed loop system is reduced to an observability estimate for the corresponding conservative adjoint system, under a boundedness condition of the transfer function of the associated open loop system.

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## 1 Introduction

In this paper, our purpose is to characterize the output feedback stabilization of coupled second order infinite dimensional systems by only one feedback. Using an output feedback, the closed loop system we treat is the following

$$\ddot{w}_1(t) + A_1 w_1(t) + BB^* \dot{w}_1(t) + C \dot{w}_2(t) = 0, \quad t \geq 0, \quad (1.1)$$

$$\ddot{w}_2(t) + A_2 w_2(t) - C^* \dot{w}_1(t) = 0, \quad t \geq 0, \quad (1.2)$$

$$w_i(0) = w_i^0, \quad \dot{w}_i(0) = w_i^1, \quad i = 1, 2. \quad (1.3)$$

Here, the operators  $A_1, A_2$  are unbounded positive self adjoint operators in Hilbert spaces  $H_1, H_2$ , respectively. The control operator  $B$ , acting only on the first equation, is assumed here to be unbounded from  $U$ , another Hilbert space, to  $D(A_1^{\frac{1}{2}})^*$ . The coupling operator  $C$  is not necessarily bounded. In [2, 3], the authors have considered coupled systems in the case of bounded (even compact) coupling operators  $C$ . In this case the exponential stability does not hold, since the equation (1.3) is conservative when  $C = 0$ . In stead, they studied the polynomial stability. Recently, Ammari and Nicaise [6] have characterized the exponential energy decay of these systems by an observability inequality of associated conservative adjoint systems, augmented with

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the output  $y(t) = -B^*\dot{w}_1(t)$ , in the case of bounded coupling operators  $C$ . In [17], the author studied also these coupled systems in the case of unbounded coupling operators, considering bounded operators  $B$ . In this paper, we assume that both operators  $B$  and  $C$  are unbounded, and show the same result as in [6] using different arguments. Here, we transform the system (1.1)-(1.3) to a second order equation

$$\ddot{w}(t) + Aw(t) + \mathcal{B}_0\mathcal{B}_0^*\dot{w}(t) = 0, \quad t \geq 0, \quad (1.4)$$

$$w(0) = w^0, \quad \dot{w}(0) = w^1 \quad (1.5)$$

in the product space  $H := H_1 \times H_2$ , with appropriate operators  $A$  and  $\mathcal{B}_0$  defined in (3.33). Then, we use the result of Ammari-Tucsnak [4, Theorem 2.2] to characterize the exponential energy decay of the equation (1.4)-(1.5), and then deduce the one of the coupled systems (1.1)-(1.3).

The second aim of this paper is to characterize the exponential energy decay of the following coupled systems with delay

$$\ddot{w}_1(t) + A_1w_1(t) + \alpha_1 BB^*\dot{w}_1(t) + \alpha_2 BB^*\dot{w}_1(t-\tau) + C\dot{w}_2(t) = 0, \quad t \geq 0, \quad (1.6)$$

$$\ddot{w}_2(t) + A_2w_2(t) - C^*\dot{w}_1(t) = 0, \quad t \geq 0, \quad (1.7)$$

$$w_i(0) = w_i^0, \quad \dot{w}_i(0) = w_i^1, \quad i = 1, 2, \quad \dot{w}_1(s) = f_0(s), \quad s \in (-\tau, 0). \quad (1.8)$$

The operators  $A_i, i = 1, 2, B, C$  satisfy the same conditions as above, and  $\alpha_1, \alpha_2$  are positive constants. The introduction of a delay term in partial differential equations and its effect on the stabilization of these equations were the subject of several papers, see for instance, [1, 7, 8, 9, 10, 11, 14, 15], and the references therein. By the same technic as for the first coupled systems, we transform the system (1.6)-(1.8) to a second order equation with delay

$$\ddot{w}(t) + Aw(t) + \alpha_1\mathcal{B}_0\mathcal{B}_0^*\dot{w}(t) + \alpha_2\mathcal{B}_0\mathcal{B}_0^*\dot{w}(t-\tau) = 0, \quad t \geq 0,$$

$$w(0) = w^0, \quad \dot{w}(0) = w^1, \quad \dot{w}_1(s) = f_0(s), \quad s \in (-\tau, 0).$$

At this level, our results in [1] will allow us to conclude.

We then apply our abstract results to two systems of coupled string equations with delay. The first example is a coupled two string equations with punctual control and Dirichlet boundary conditions

$$\begin{aligned} \ddot{w}_1(t, x) - \frac{\partial^2 w_1}{\partial x^2}(t, x) + \alpha_1 \dot{w}_1(t, \xi) \delta_\xi + \alpha_2 \dot{w}_1(t-\tau, \xi) \delta_\xi + \beta \frac{\partial \dot{w}_2}{\partial x}(t, x) &= 0, \quad (t, x) \in (0, \infty) \times (0, 1), \\ \ddot{w}_2(t, x) - \frac{\partial^2 w_2}{\partial x^2}(t, x) + \beta \frac{\partial \dot{w}_1}{\partial x}(t, x) &= 0, \quad (t, x) \in (0, \infty) \times (0, 1), \\ w_i(t, 0) = w_i(t, 1) &= 0, \quad t \in (0, \infty), \quad i = 1, 2, \\ w_i(0, x) = w_i^0(x), \quad \dot{w}_i(0, x) = w_i^1(x), \quad \dot{w}_1(s, x) &= f_0(s, x), \quad -\tau \leq s < 0, \quad x \in (0, 1), \quad i = 1, 2, \end{aligned}$$

with  $\xi \in (0, 1)$ ,  $\beta > 0$  and  $0 < \alpha_2 < \alpha_1$ . We show that this system is not exponentially stable for all  $\xi \in (0, 1)$  and  $\beta > 0$ , showing that the observability inequality of its conservative adjoint system can not hold. To give a positive application of our abstract results, we consider a coupled two wave equations with punctual control and mixed boundary conditions

$$\ddot{w}_1(t, x) - \frac{\partial^2 w_1}{\partial x^2}(t, x) + w_1(t, x) + \alpha_1 \dot{w}_1(t, \xi) \delta_\xi + \alpha_2 \dot{w}_1(t-\tau, \xi) \delta_\xi + \beta \frac{\partial \dot{w}_2}{\partial x}(t, x) = 0, \quad t \geq 0, \quad x \in (0, 1),$$

$$\begin{aligned} \ddot{w}_2(t, x) - \frac{\partial^2 w_2}{\partial x^2}(t, x) + w_2(t, x) + \beta \frac{\partial \dot{w}_1}{\partial x}(t, x) &= 0, \quad t \geq 0, x \in (0, 1), \\ \frac{\partial w_1}{\partial x}(t, 0) &= \frac{\partial w_1}{\partial x}(t, 1) = 0, \quad w_2(t, 0) = w_2(t, 1) = 0, \quad t \geq 0, \\ w_i(0, x) &= w_i^0(x), \quad \dot{w}_i(0, x) = w_i^1(x), \quad \dot{w}_1(s, x) = f_0(s, x), \quad -\tau \leq s < 0, x \in (0, 1), \quad i = 1, 2 \end{aligned}$$

with  $\xi \in (0, 1)$ ,  $\beta$  is a positive constant and  $0 < \alpha_2 < \alpha_1$ . Using the classical inequality by Ingham [13] for non-harmonic Fourier series, we show that the observability inequality of the conservative adjoint system holds if and only if  $\xi$  is a rational number with coprime factorisation  $\xi = \frac{p}{q}$ , where  $p$  is odd. Thus, this is a necessary and sufficient condition for the exponential energy decay of the above system.

## 2 Problem formulation

Let  $H_i$  be a Hilbert space equipped with the norm  $\|\cdot\|_{H_i}, i = 1, 2$  and let

$$A_i : H_i \supseteq \mathcal{D}(A_i) \rightarrow H_i, i = 1, 2, \text{ be positive self adjoint operators.} \quad (2.9)$$

We introduce the scale of Hilbert spaces  $H_{i,\alpha}$  as  $H_{i,\alpha} = \mathcal{D}(A_i^\alpha)$  with the norm  $\|z\|_{i,\alpha} = \|A_i^\alpha z\|_{H_i}$  and their dual spaces  $H_{i,-\alpha} = H_{i,\alpha}^*, i = 1, 2$ . The second ingredient needed for our construction is a control operator  $B$  such that

$$B : U \longrightarrow H_{1,-\frac{1}{2}} \text{ is bounded,} \quad (2.10)$$

where  $U$  is another Hilbert space identified with its dual. The operator  $B^*$  is then bounded from  $H_{1,\frac{1}{2}}$  to  $U$ . We need also a unbounded linear operator  $C : H_2 \supseteq \mathcal{D}(C) \longrightarrow H_1$  satisfying the following assumptions

$$H_{1,\frac{1}{2}} \hookrightarrow \mathcal{D}(C^*) \text{ and } H_{2,\frac{1}{2}} \hookrightarrow \mathcal{D}(C). \quad (2.11)$$

**Remark 2.1.** By assumptions (2.11), one can see that the operators  $CA_2^{-\frac{1}{2}}$  and  $C^*A_1^{-\frac{1}{2}}$  can be extended to bounded operators from  $H_2$  to  $H_1$ .

The first coupled systems that we consider are described by

$$\ddot{w}_1(t) + A_1 w_1(t) + BB^* \dot{w}_1(t) + C \dot{w}_2(t) = 0, \quad t \geq 0, \quad (2.12)$$

$$\ddot{w}_2(t) + A_2 w_2(t) - C^* \dot{w}_1(t) = 0, \quad t \geq 0, \quad (2.13)$$

$$w_i(0) = w_i^0, \quad \dot{w}_i(0) = w_i^1, \quad i = 1, 2, \quad (2.14)$$

where the initial data  $(w_1^0, w_1^1, w_2^0, w_2^1)$  belongs to a suitable space.

The equation (2.12) is understood as equation in  $H_{1,-\frac{1}{2}}$ , i.e., all the terms are in  $H_{1,-\frac{1}{2}}$ . The term  $BB^* \dot{w}_1(t)$  represents a feedback damping. Transforming system (2.12)-(2.14) on a second order system and using the method in [4], we characterize the stabilization of this system. Namely, assuming that there exists  $\delta \in [0, \frac{1}{2})$  such that for all  $(x, y) \in H_{1,1} \times H_{2,1}$

$$| \langle x, Cy \rangle | \leq \delta \left( \|A_1^{\frac{1}{2}} x\|^2 + \|y\|^2 + \|C^* x\|^2 \right), \quad (2.15)$$

under the boundedness of corresponding transfer function, system (2.12)-(2.14) is exponentially stable if and only if there exists a constant  $c > 0$  such that

$$c \int_0^T \| (B^* \phi)' \|_U^2 dt \geq \| A_1^{\frac{1}{2}} \phi(0) \|^2 + \| A_2^{\frac{1}{2}} \psi(0) \|^2 + \| \begin{pmatrix} \dot{\phi}(0) \\ \dot{\psi}(0) \end{pmatrix} \|_{H_1 \times H_2}^2 \quad (2.16)$$

for all solution  $(\phi, \psi)$  of the following conservative adjoint system

$$\begin{aligned} \ddot{\phi} + A_1 \phi + C \dot{\psi} &= 0 \\ \ddot{\psi} + A_2 \psi - C^* \dot{\phi} &= 0. \end{aligned}$$

Our second interest is to characterize the stabilization of the following coupled systems with delay

$$\ddot{w}_1(t) + A_1 w_1(t) + \alpha_1 BB^* \dot{w}_1(t) + \alpha_2 BB^* \dot{w}_1(t-\tau) + C \dot{w}_2(t) = 0, \quad t \geq 0, \quad (2.17)$$

$$\ddot{w}_2(t) + A_2 w_2(t) - C^* \dot{w}_1(t) = 0, \quad t \geq 0, \quad (2.18)$$

$$w_i(0) = w_i^0, \dot{w}_i(0) = w_i^1, i = 1, 2, \dot{w}_1(s) = f_0(s), s \in (-\tau, 0), \quad (2.19)$$

where  $\tau > 0$  is the time delay,  $\alpha_1$  and  $\alpha_2$  are positive real numbers, and the initial data  $(w_1^0, w_1^1, w_2^0, w_2^1, f_0)$  belongs to a suitable space. Assuming that  $\alpha_2 < \alpha_1$ , under the same assumption (2.15) we prove that (2.17)-(2.19) is exponentially stable if and only if the observability inequality (2.16) is satisfied, which is then equivalent to the exponential stability of (2.12)-(2.14).

### 3 Coupled second order systems without delay

Consider the following coupled systems

$$\ddot{w}_1(t) + A_1 w_1(t) + BB^* \dot{w}_1(t) + C \dot{w}_2(t) = 0, \quad t \geq 0, \quad (3.20)$$

$$\ddot{w}_2(t) + A_2 w_2(t) - C^* \dot{w}_1(t) = 0, \quad t \geq 0, \quad (3.21)$$

$$w_i(0) = w_i^0, \dot{w}_i(0) = w_i^1, i = 1, 2. \quad (3.22)$$

After studying the well-posedness of the coupled systems (3.20)-(3.22), we give a characterization of its exponential stability.

#### 3.1 Well-posedness

Some change of variables, leads to the following result

**Theorem 3.1.** *If  $(w_1, w_2)$  is a solution of (3.20)-(3.22), then  $(u, v)$  defined by*

$$u = w_1, \quad v = A_2^{-\frac{1}{2}} \dot{w}_2 - A_2^{-\frac{1}{2}} C^* w_1$$

*is a solution of the system*

$$\ddot{u}(t) + (A_1 + CC^*) u(t) + BB^* \dot{u}(t) + CA_2^{\frac{1}{2}} v(t) = 0, \quad t \geq 0, \quad (3.23)$$

$$\ddot{v}(t) + A_2 v(t) + A_2^{\frac{1}{2}} C^* u(t) = 0, \quad t \geq 0. \quad (3.24)$$

$$u(0) = u^0, \dot{u}(0) = u^1, v(0) = v^0, \dot{v}(0) = v^1 \quad (3.25)$$

with  $u^0 = w_1^0, u^1 = w_1^1, v^0 = A_2^{-\frac{1}{2}} w_2^1 - A_2^{-\frac{1}{2}} C^* w_1^0, v^1 = -A_2^{\frac{1}{2}} w_2^0$ .

Conversely, if  $(u, v)$  is a solution of (3.23)-(3.25), then  $(w_1, w_2)$  defined by

$$w_1 = u, \quad w_2 = -A_2^{\frac{1}{2}} \dot{v}$$

is a solution of (3.20)-(3.22).

*Proof.* Let  $(w_1, w_2)$  be a solution of (3.20)-(3.22). Setting  $u = w_1$  and  $v = A_2^{-\frac{1}{2}} \dot{w}_2 - A_2^{-\frac{1}{2}} C^* w_1$ , we have

$$\begin{aligned} u(t) &= w_1(t), \quad v(t) = A_2^{-\frac{1}{2}} \dot{w}_2(t) - A_2^{-\frac{1}{2}} C^* w_1(t), \quad t \geq 0, \\ \dot{u}(t) &= \dot{w}_1(t), \quad \dot{v}(t) = A_2^{-\frac{1}{2}} \ddot{w}_2(t) - A_2^{-\frac{1}{2}} C^* \dot{w}_1(t), \quad t \geq 0. \end{aligned}$$

Equation (3.21) yields

$$u(t) = w_1(t), \quad v(t) = A_2^{-\frac{1}{2}} \dot{w}_2(t) - A_2^{-\frac{1}{2}} C^* w_1(t), \quad \dot{u}(t) = \dot{w}_1(t), \quad t \geq 0, \quad \dot{v}(t) = -A_2^{-\frac{1}{2}} w_2(t), \quad t \geq 0.$$

Thus

$$\begin{pmatrix} u(t) \\ v(t) \\ \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} = P \begin{pmatrix} w_1(t) \\ w_2(t) \\ \dot{w}_1(t) \\ \dot{w}_2(t) \end{pmatrix}, \quad t \geq 0, \quad (3.26)$$

where

$$P = \begin{pmatrix} I & 0 & 0 & 0 \\ -A_2^{-\frac{1}{2}} C^* & 0 & 0 & A_2^{-\frac{1}{2}} \\ 0 & 0 & I & 0 \\ 0 & -A_2^{\frac{1}{2}} & 0 & 0 \end{pmatrix}.$$

Together with (3.21), derivation of the equation (3.26) leads to the coupled systems (3.23)-(3.24). The initial data (3.25) follows from (3.26).

By Remark 2.1,  $P$  is a bounded and invertible operator from  $\mathcal{H} := H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}} \times H_1 \times H_2$  to  $\mathcal{H}$  with inverse

$$P^{-1} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_2^{-\frac{1}{2}} \\ 0 & 0 & I & 0 \\ C^* & A_2^{\frac{1}{2}} & 0 & 0 \end{pmatrix}.$$

Using  $P^{-1}$ , the converse in Theorem 3.1 can be similarly proved.  $\square$

The equivalence of the well-posedness of the systems (3.20)-(3.21) and (3.23)-(3.25) can be proved also by using their corresponding Cauchy problems. Roughly speaking, setting  $X := \begin{pmatrix} w_1 \\ w_2 \\ \dot{w}_1 \\ \dot{w}_2 \end{pmatrix}$ , the system (3.20)-(3.21) can be transformed in  $\mathcal{H}$  to the following first order system

$$\dot{X} = \mathcal{A}_1 X, \quad X(0) = \begin{pmatrix} w_1^0 \\ w_2^0 \\ \dot{w}_1^0 \\ \dot{w}_2^0 \end{pmatrix}, \quad (3.27)$$

where

$$\mathcal{A}_1 : \mathcal{D}(\mathcal{A}_1) \subset \mathcal{H} \longrightarrow \mathcal{H}, \quad \mathcal{A}_1 \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -A_1 u_1 - BB^* u_2 - Cv_2 \\ v_2 \\ -A_2 v_1 + C^* u_2 \end{pmatrix}, \quad (3.28)$$

and

$$\mathcal{D}(\mathcal{A}_1) := \left\{ (u_1, v_1, u_2, v_2) \in H_{1,\frac{1}{2}} \times H_{2,1} \times H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}}, \quad A_1 u_1 + BB^* u_2 \in H_1 \right\}.$$

The system (3.23)-(3.25) can be written as

$$\dot{Y} = \mathcal{A}_2 Y, \quad Y(0) = \begin{pmatrix} u^0 \\ v^0 \\ u^1 \\ v^1 \end{pmatrix}, \quad (3.29)$$

where

$$\mathcal{A}_2 : \mathcal{D}(\mathcal{A}_2) \subset \mathcal{H} \longrightarrow \mathcal{H}, \quad \mathcal{A}_2 \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ v_2 \\ -A_1 u_1 - C(C^* u_1 + A_2^{\frac{1}{2}} v_1) - BB^* u_2 \\ -A_2^{\frac{1}{2}} (C^* u_1 + A_2^{\frac{1}{2}} v_1) \end{pmatrix}, \quad (3.30)$$

and

$$\mathcal{D}(\mathcal{A}_2) := \left\{ (u_1, v_1, u_2, v_2) \in H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}} \times H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}}, \quad A_1 u_1 + BB^* u_2 \in H_1, \quad C^* u_1 + A_2^{\frac{1}{2}} v_1 \in H_{2,\frac{1}{2}} \right\}.$$

For every  $\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \in \mathcal{D}(\mathcal{A}_1)$ , we have

$$\begin{aligned} P \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} &= \begin{pmatrix} u_1 \\ -A_2^{-\frac{1}{2}} Cu_1 + A_2^{-\frac{1}{2}} v_2 \\ u_2 \\ -A_2^{\frac{1}{2}} v_1 \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ -A_2^{-1} A_2^{\frac{1}{2}} Cu_1 + A_2^{-\frac{1}{2}} v_2 \\ u_2 \\ -A_2^{\frac{1}{2}} v_1 \end{pmatrix}. \end{aligned}$$

Since  $C^* u_1 + A_2^{\frac{1}{2}} (-A_2^{-\frac{1}{2}} Cu_1 + A_2^{-\frac{1}{2}} v_2) = v_2 \in H_{2,\frac{1}{2}}$  and  $A_1 u_1 + BB^* u_2 \in H_1$ , we have  $P\mathcal{D}(\mathcal{A}_1) \subset \mathcal{D}(\mathcal{A}_2)$ . Using (3.26), we can see that  $\mathcal{A}_1 = P^{-1} \mathcal{A}_2 P$ .

To study the well-posedness and exponential stability of both coupled systems, we write the system (3.23)-(3.25), in the product space  $H := H_1 \times H_2$ , as the following second order system

$$\ddot{W}(t) + AW(t) + \mathcal{B}_0 \mathcal{B}_0^* \dot{W}(t) = 0, \quad t \geq 0, \quad (3.31)$$

$$W(0) = W^0, \quad \dot{W}(0) = W^1, \quad (3.32)$$

where

$$A : \mathcal{D}(A) \subset H \longrightarrow H, \quad A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_1 u + C(C^* u + A_2^{\frac{1}{2}} v) \\ A_2^{\frac{1}{2}} (C^* u + A_2^{\frac{1}{2}} v) \end{pmatrix}, \quad \mathcal{B}_0 = \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad (3.33)$$

with  $\mathcal{D}(A) = \{(u, v) \in H_{1,1} \times H_{2,\frac{1}{2}}, \quad C^* u + A_2^{\frac{1}{2}} v \in H_{2,\frac{1}{2}}\}$ .

To obtain the well-posedness result, we need the following lemma which will be also crucial for the rest of this paper.

**Lemma 3.2.** *The following assertions hold.*

- (i) *The operator  $A$  is positive self adjoint.*
- (ii)  $\mathcal{B}_0^* : \mathcal{D}(A^{\frac{1}{2}}) \rightarrow U$  *is a bounded operator.*
- (iii)  $\mathcal{B}_0 : U \rightarrow \mathcal{D}(A^{\frac{1}{2}})^* := H_{\frac{1}{2}}$  *is a bounded operator.*

*Proof.* (i) Let  $(\begin{smallmatrix} x \\ y \end{smallmatrix}) \in H_{1,1} \times H_{2,1}$ . We have

$$\langle A(\begin{smallmatrix} x \\ y \end{smallmatrix}), (\begin{smallmatrix} x \\ y \end{smallmatrix}) \rangle = \|A_1^{\frac{1}{2}}x\|^2 + \langle C^*x + A_2^{\frac{1}{2}}y, C^*x \rangle + \langle C^*x + A_2^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y \rangle.$$

Then

$$\langle A(\begin{smallmatrix} x \\ y \end{smallmatrix}), (\begin{smallmatrix} x \\ y \end{smallmatrix}) \rangle = \|A_1^{\frac{1}{2}}x\|^2 + \|A_2^{\frac{1}{2}}y + C^*x\|^2 > 0.$$

Thus,  $A$  is a symmetric positive operator. For every  $(f, g) \in H$ , the solution  $(u, v) \in \mathcal{D}(A)$  of the system

$$\begin{aligned} A_1u + C(C^*u + A_2^{\frac{1}{2}}v) &= f, \\ A_2^{\frac{1}{2}}(C^*u + A_2^{\frac{1}{2}}v) &= g \end{aligned}$$

is given by

$$u = A_1^{-1}(f - CA_2^{-\frac{1}{2}}g), \quad v = A_2^{-1}g - A_2^{-\frac{1}{2}}C^*A_1^{-1}(f - CA_2^{-\frac{1}{2}}g).$$

It is clear that  $(u, v) \in H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}}$ . Since  $C^*u + A_2^{\frac{1}{2}}v = A_2^{-\frac{1}{2}}g \in H_{2,\frac{1}{2}}$ , we have  $(u, v) \in \mathcal{D}(A)$ . Thus, the operator  $A$  is invertible. Consequently,  $A$  is a positive self adjoint operator.

(ii) Let  $(\begin{smallmatrix} x \\ y \end{smallmatrix}) \in \mathcal{D}(A^{\frac{1}{2}})$ . We have  $\mathcal{B}_0^*(\begin{smallmatrix} x \\ y \end{smallmatrix}) = B^*x$ . Since  $B^*$  is a bounded operator from  $H_{1,\frac{1}{2}}$  to  $U$ , there exists a constant  $c > 0$  such that  $\|B^*x\|_U \leq c\|A_1^{\frac{1}{2}}x\|_{H_1}$ . Thus,

$$\|\mathcal{B}_0^*(\begin{smallmatrix} x \\ y \end{smallmatrix})\|_U \leq c[\|A_1^{\frac{1}{2}}x\|^2 + \|A_2^{\frac{1}{2}}y + C^*x\|^2],$$

and thus the operator  $\mathcal{B}_0^* : \mathcal{D}(A^{\frac{1}{2}}) \rightarrow U$  is bounded. The assertion (iii) follows from (ii).  $\square$

As a consequence of the above lemma we have the following well-posedness result.

**Proposition 3.3.** *Assume that (2.9), (2.10) and (2.11) hold. Then, the system*

$$\ddot{W}(t) + AW(t) + \mathcal{B}_0\mathcal{B}_0^*\dot{W}(t) = 0, \quad t \geq 0, \tag{3.34}$$

$$W(0) = W^0, \quad \dot{W}(0) = W^1 \tag{3.35}$$

is well-posed in the energy space  $\mathcal{D}(A^{\frac{1}{2}}) \times H$ .

Using Theorem 3.1, Proposition 3.3 and the regularity results in [4], we have the following well-posedness and regularity result of the coupled systems (3.20)-(3.22).

**Proposition 3.4.** *Assume that (2.9), (2.10) and (2.11) hold. Then, the system (3.20)-(3.22) is well-posed, i.e.,*

- (i) *for  $(w_1^0, w_2^0, w_1^1, w_2^1) \in \mathcal{D}(\mathcal{A}_1)$ , the problem (3.20)-(3.22) admits a unique solution  $w_i \in C^1([0, T]; H_{i,\frac{1}{2}}) \cap C^2([0, T]; H_i)$ ,  $i = 1, 2$ ,*
- (ii) *for  $(w_1^0, w_2^0, w_1^1, w_2^1) \in \mathcal{H}$ ,  $w_i \in C([0, T]; H_{i,\frac{1}{2}}) \cap C^1([0, T]; H_i)$ ,  $i = 1, 2$ , and  $B^*w_1(\cdot) \in H^1(0, T; U)$ .*

**Remark 3.5.** *The well-posedness of (3.20)-(3.22) can be also obtained directly by proving that the operator  $\mathcal{A}_1$  satisfies the conditions of Lumer-Phillips theorem, see [12].*

### 3.2 Transfer function

To characterize the stabilization of system (3.20)-(3.22) we need the following lemma.

**Lemma 3.6.** *Assume that (2.9), (2.10) and (2.11) hold. Then, the following results hold.*

(i) *The operator  $[\lambda^2 + A_1 + \lambda^2 C(\lambda^2 + A_2)^{-1} C^*]$  is invertible from  $H_{1,\frac{1}{2}}$  to  $H_{1,-\frac{1}{2}}$ .*

(ii) *The function defined by*

$$G(\lambda) = \lambda B^* [\lambda^2 + A_1 + \lambda^2 C(\lambda^2 + A_2)^{-1} C^*]^{-1} B, \quad \lambda > 0,$$

*is the transfer function of both systems (3.20)-(3.21) and (3.34)-(3.35).*

*Proof.* (i) Let  $y \in H_{1,-\frac{1}{2}}$ . Consider in  $H_{1,\frac{1}{2}}$  the equation

$$[\lambda^2 + A_1 + \lambda^2 C(\lambda^2 + A_2)^{-1} C^*]x = y. \quad (3.36)$$

For every  $\zeta \in H_{1,\frac{1}{2}}$ , we have

$$\langle [\lambda^2 + A_1 + \lambda^2 C(\lambda^2 + A_2)^{-1} C^*]x, \zeta \rangle = \langle y, \zeta \rangle$$

which can be written as

$$\lambda^2 \langle x, \zeta \rangle + \left\langle A_1^{\frac{1}{2}} x, A_1^{\frac{1}{2}} \zeta \right\rangle + \left\langle \lambda(\lambda^2 + A_2)^{-\frac{1}{2}} C^* x, \lambda(\lambda^2 + A_2)^{-\frac{1}{2}} C^* \zeta \right\rangle = \langle y, \zeta \rangle =: \Lambda(x, \zeta).$$

Since  $\Lambda$  is a bilinear coercive form on  $H_{1,\frac{1}{2}}$ , the Lax-Milgram theorem leads to the existence and uniqueness of the solution  $x$  to the equation (3.36), and thus the claim follows.

(ii) We compute first the transfer function of (3.34)-(3.35). Setting  $Z := \begin{pmatrix} W \\ \dot{W} \end{pmatrix}$ , the open loop system associated to (3.34)-(3.35) can be transformed to the following controlled first order system in the energy space  $\mathcal{D}(A^{\frac{1}{2}}) \times H$

$$\dot{Z}(t) = \mathcal{A}_2^0 Z(t) + \mathcal{B} u(t), \quad t \geq 0, \quad (3.37)$$

$$Z(0) = \begin{pmatrix} W^0 \\ W^1 \end{pmatrix}, \quad (3.38)$$

with  $\mathcal{A}_2^0 = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$ ,  $\mathcal{D}(\mathcal{A}_2^0) = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}})$ , and  $\mathcal{B} = \begin{pmatrix} \mathcal{B}_0 \\ 0 \end{pmatrix}$ .

Let  $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(A^{\frac{1}{2}}) \times H$ . We look for  $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}})$  such that

$$(\lambda - \mathcal{A}_2^0) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \quad (3.39)$$

We have

$$\begin{aligned} (3.39) \iff & \begin{cases} \lambda x - y = f \\ \lambda y + Ax = g \end{cases} \\ \iff & \begin{cases} x = \lambda(\lambda^2 + A)^{-1}f + (\lambda^2 + A)^{-1}g \\ y = (\lambda^2(\lambda^2 + A)^{-1} - I)f + \lambda(\lambda^2 + A)^{-1}g, \end{cases} \end{aligned}$$

and thus

$$(\lambda - \mathcal{A}_2^0)^{-1} = \begin{pmatrix} \lambda(\lambda^2 + A)^{-1} & (\lambda^2 + A)^{-1} \\ (\lambda^2(\lambda^2 + A)^{-1} - I) & \lambda(\lambda^2 + A)^{-1} \end{pmatrix}.$$

The transfer function  $G_2(\lambda) := \mathcal{B}^*(\lambda - \mathcal{A}_2^0)^{-1}\mathcal{B}$  of the system (3.37)-(3.38) is then

$$G_2(\lambda) = (\mathcal{B}_0^* \quad 0) \begin{pmatrix} \lambda(\lambda^2 + A)^{-1}\mathcal{B}_0 \\ (\lambda^2(\lambda^2 + A)^{-1} - I)\mathcal{B}_0 \end{pmatrix} = \lambda\mathcal{B}_0^*(\lambda^2 + A)^{-1}\mathcal{B}_0.$$

Easy computation leads to

$$(\lambda^2 + A)^{-1} = \begin{pmatrix} \Gamma & -\Gamma C A_2^{\frac{1}{2}}(\lambda^2 + A_2)^{-1} \\ -(\lambda^2 + A_2)^{-1} A_2^{\frac{1}{2}} C^* \Gamma & (\lambda^2 + A_2)^{-1} [I + A_2^{\frac{1}{2}} C^* \Gamma C A_2^{\frac{1}{2}}(\lambda^2 + A_2)^{-1}] \end{pmatrix},$$

where  $\Gamma := [\lambda^2 + A_1 + \lambda^2 C(\lambda^2 + A_2)^{-1} C^*]^{-1}$ . Consequently,

$$G_2(\lambda) = \lambda B^*[\lambda^2 + A_1 + \lambda^2 C(\lambda^2 + A_2)^{-1} C^*]^{-1} B, \quad \forall \lambda > 0.$$

Let  $\mathcal{A}_1^0 : \mathcal{D}(\mathcal{A}_1^0) \subset \mathcal{H} \longrightarrow \mathcal{H}$ ,  $\mathcal{A}_1^0 \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ v_2 \\ -A_1 u_1 - C v_2 \\ -A_2 v_1 + C^* u_2 \end{pmatrix}$  be the generator of the open loop system associated to (3.20)-(3.22). Since  $\mathcal{A}_1^0 = P^{-1}\mathcal{A}_2^0P$ , we have

$$(\lambda - \mathcal{A}_1^0)^{-1} = P^{-1}(\lambda - \mathcal{A}_2^0)^{-1}P, \quad \forall \lambda > 0.$$

Since  $\mathcal{B}_0^*P^{-1} = \mathcal{B}_0^*$  and  $P\mathcal{B}_0 = \mathcal{B}_0$ , we have

$$G_1(\lambda) := \mathcal{B}_0^*(\lambda - \mathcal{A}_1^0)^{-1}\mathcal{B}_0 = \mathcal{B}_0^*(\lambda - \mathcal{A}_2^0)^{-1}\mathcal{B}_0 = G_2(\lambda).$$

### 3.3 Stabilization

In order to characterize the stabilization of the coupled systems without delay, we give some energy equivalences.

**Proposition 3.7.** *Assume that (2.9), (2.10), (2.11) and (2.15) hold. Then,*

$$\begin{aligned} \mathcal{E}(t) &:= \frac{1}{2} \left( \|A^{\frac{1}{2}} \begin{pmatrix} u \\ v \end{pmatrix}\|^2 + \left\| \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \right\|_{H_1 \times H_2}^2 \right) \\ &\asymp \\ \widetilde{E}(t) &:= \frac{1}{2} \left( \|A_1^{\frac{1}{2}} u\|^2 + \|A_2^{\frac{1}{2}} v\|^2 + \|C^* u\|^2 + \left\| \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \right\|_{H_1 \times H_2}^2 \right) \\ &\asymp \\ E(t) &:= \frac{1}{2} \left( \|A_1^{\frac{1}{2}} w_1\|^2 + \|A_2^{\frac{1}{2}} w_2\|^2 + \left\| \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} \right\|_{H_1 \times H_2}^2 \right), \end{aligned}$$

for every solutions  $(w_1, w_2)$  and  $(u, v)$  of (3.20)-(3.22) and (3.23)-(3.25), respectively.

From this follows immediately the following corollary.

**Corollary 3.8.** *Assume that (2.9), (2.10), (2.11) and (2.15) hold. Then, The exponential stabilities of the three systems (3.20)-(3.22), (3.23)-(3.25), and (3.34)-(3.35) are equivalent.*

Using the characterization of stabilization of second order equation in [4], we have the following result.

**Theorem 3.9.** *Assume that (2.9), (2.10), (2.11) and (2.15) hold and for a fixed  $\gamma > 0$*

$$\sup_{\operatorname{Re}\lambda=\gamma} \left\| \lambda B^* [\lambda^2 I + A_1 + \lambda^2 C(\lambda^2 + A_2)^{-1} C^*]^{-1} B \right\|_{\mathcal{L}(U)} < \infty. \quad (3.40)$$

*The system (3.34)-(3.35) is exponentially stable in  $\mathcal{D}(A^{\frac{1}{2}}) \times H$  if and only if there exists a constant  $c > 0$  such that*

$$c \int_0^T \| (B^* \phi_1)' \|_U^2 dt \geq \| A^{\frac{1}{2}} \begin{pmatrix} \phi_1(0) \\ \psi_1(0) \end{pmatrix} \|^2 + \| \begin{pmatrix} \dot{\phi}_1(0) \\ \dot{\psi}_1(0) \end{pmatrix} \|_{H_1 \times H_2}^2, \quad (3.41)$$

where  $(\phi_1, \psi_1)$  is a solution of the following system

$$\ddot{\phi}_1 + (A_1 + CC^*)\phi_1 + CA_2^{\frac{1}{2}}\psi_1 = 0 \quad (3.42)$$

$$\ddot{\psi}_1 + A_2\psi_1 + A_2^{\frac{1}{2}}C^*\phi_1 = 0. \quad (3.43)$$

As a consequence of the above theorem, we have the following result.

**Theorem 3.10.** *Assume that (2.9), (2.10), (2.11), (2.15), and (3.40) hold. Then the following assertions are equivalent.*

- (i) *The system (3.20)-(3.22) is exponentially stable in  $\mathcal{H}$ .*
- (ii) *There exists a constant  $c > 0$  such that*

$$c \int_0^T \| (B^* \phi)' \|_U^2 dt \geq \| A_1^{\frac{1}{2}} \phi(0) \|^2 + \| A_2^{\frac{1}{2}} \psi(0) \|^2 + \| C^* \phi(0) \|^2 + \| \begin{pmatrix} \dot{\phi}(0) \\ \dot{\psi}(0) \end{pmatrix} \|_{H_1 \times H_2}^2. \quad (3.44)$$

- (iii) *There exists a constant  $c > 0$  such that*

$$c \int_0^T \| (B^* \phi)' \|_U^2 dt \geq \| A_1^{\frac{1}{2}} \phi(0) \|^2 + \| A_2^{\frac{1}{2}} \psi(0) \|^2 + \| \begin{pmatrix} \dot{\phi}(0) \\ \dot{\psi}(0) \end{pmatrix} \|_{H_1 \times H_2}^2, \quad (3.45)$$

where  $(\phi, \psi)$  is a solution of the following conservative adjoint system

$$\ddot{\phi} + A_1\phi + C\dot{\psi} = 0, \quad (3.46)$$

$$\ddot{\psi} + A_2\psi - C^*\dot{\phi} = 0. \quad (3.47)$$

*Proof.* From Corollary 3.8 and Theorem 3.9, the assertion (i) is equivalent to the observability inequality (3.41). To show (3.44) and (3.45), let  $(\phi_1, \psi_1)$  be a solution of (3.42)-(3.43). Then  $\phi = \phi_1$  and  $\psi = A_2^{-\frac{1}{2}}\dot{\psi}_1$  satisfy (3.46)-(3.47). The observability inequality (3.41) becomes

$$c \int_0^T \| (B^* \phi)' \|_U^2 dt \geq \| A_1^{\frac{1}{2}} \phi(0) \|^2 + \| A_2^{\frac{1}{2}} \psi(0) \|^2 + \| C^* \phi(0) \|^2 + \| \begin{pmatrix} \dot{\phi}(0) \\ \dot{\psi}(0) \end{pmatrix} \|_{H_1 \times H_2}^2 + 2\operatorname{Re} \langle \phi(0), C\dot{\psi}(0) \rangle.$$

Since by (2.15),

$$| \langle \phi, C\dot{\psi}(0) \rangle | < \delta \left( \| A_1^{\frac{1}{2}} \phi(0) \|^2 + \| \dot{\psi}(0) \|^2 + \| C^* \phi(0) \|^2 \right)$$

with  $\delta \in [0, \frac{1}{2})$ , the inequality (3.41) can be written as

$$c \int_0^T \| (B^* \phi)' \|_U^2 dt \geq \| A_1^{\frac{1}{2}} \phi(0) \|^2 + \| A_2^{\frac{1}{2}} \psi(0) \|^2 + \| C^* \phi(0) \|^2 + \| \begin{pmatrix} \dot{\phi}(0) \\ \dot{\psi}(0) \end{pmatrix} \|_{H_1 \times H_2}^2,$$

which is exactly the inequality (3.44). Now from the assumption  $H_{1, \frac{1}{2}} \hookrightarrow \mathcal{D}(C^*)$  follows the inequality (3.45). The converse can be shown in the same way.  $\square$

## 4 Coupled second order systems with delay

Consider in this section the following coupled systems with delay

$$\ddot{w}_1(t) + A_1 w_1(t) + \alpha_1 BB^* \dot{w}_1(t) + \alpha_2 BB^* \dot{w}_1(t - \tau) + C \dot{w}_2(t) = 0, \quad t \geq 0, \quad (4.48)$$

$$\ddot{w}_2(t) + A_2 w_2(t) - C^* \dot{w}_1(t) = 0, \quad t \geq 0, \quad (4.49)$$

$$w_i(0) = w_i^0, \dot{w}_i(0) = w_i^1, i = 1, 2, \dot{w}_1(s) = f_0(s), s \in (-\tau, 0), \quad (4.50)$$

where  $\tau > 0$  is the time delay,  $\alpha_1$  and  $\alpha_2$  are positive real numbers, and the initial data  $(w_1^0, w_1^1, w_2^0, w_2^1, f_0)$  belongs to a suitable space.

Using the same method as in the coupled systems without delay, the system (4.48)-(4.49) can be transformed to the following one

$$\ddot{u}(t) + (A_1 + CC^*)u(t) + \alpha_1 BB^* \dot{u}(t) + \alpha_2 BB^* \dot{u}(t - \tau) + CA_2^{\frac{1}{2}}v(t) = 0, \quad t \geq 0, \quad (4.51)$$

$$\ddot{v}(t) + A_2 v(t) + A_2^{\frac{1}{2}}C^* u(t) = 0, \quad t \geq 0. \quad (4.52)$$

$$u(0) = u^0, \dot{u}(0) = u^1, v(0) = v^0, \dot{v}(0) = v^1, \dot{u}(s) = f_0(s), s \in (-\tau, 0), \quad (4.53)$$

with  $u^0 := w_1^0, u^1 := w_1^1, v^0 := A_2^{-\frac{1}{2}}w_2^1 - A_2^{-\frac{1}{2}}C^*w_1^0, v^1 := -A_2^{\frac{1}{2}}w_2^0$ . This system can be written in the space  $H = H_1 \times H_2$  under the following second order system with delay

$$\ddot{W}(t) + AW(t) + \alpha_1 \mathcal{B}_0 \mathcal{B}_0^* \dot{W}(t) + \alpha_2 \mathcal{B}_0 \mathcal{B}_0^* \dot{W}(t - \tau) = 0, \quad (4.54)$$

$$W(0) = W^0, \dot{W}(0) = W^1, \dot{W}(s) = \begin{pmatrix} f_0(s) \\ 0 \end{pmatrix}, s \in (-\tau, 0), \quad (4.55)$$

with  $A$  and  $\mathcal{B}_0$  are defined in the previous section. Let  $E_{1,\frac{1}{2}}$  be the topological supplement of  $\ker B^*$  in  $H_{1,\frac{1}{2}}$  and  $P_2$  its associated projection. It is clear that  $E_{1,\frac{1}{2}} \times \{0\}$  is the topological supplement of  $\ker \mathcal{B}_0^*$  in  $H_{\frac{1}{2}} = H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}}$  and the associated projection  $\mathcal{P}_2$  is given by  $\mathcal{P}_2 W^0 = \begin{pmatrix} P_2 u^0 \\ 0 \end{pmatrix}$ . As in [1], the second order equation with delay (4.54)-(4.55) can be written as the Cauchy problem

$$\dot{\tilde{Z}}(t) = \mathcal{A}_d \tilde{Z}(t), \quad t \geq 0, \quad \tilde{Z}(0) = \begin{pmatrix} W^0 \\ W^1 \\ P_2 f_0 \end{pmatrix} \quad (4.56)$$

in the Hilbert space  $H_{\frac{1}{2}} \times H \times L^2(-\tau, 0; \mathcal{P}_2 H_{\frac{1}{2}})$  which can be identified with  $\tilde{\mathcal{H}} := H_{\frac{1}{2}} \times H \times L^2(-\tau, 0; E_{1,\frac{1}{2}})$ , where  $\tilde{Z} = \begin{pmatrix} u \\ v \\ \dot{u} \\ \dot{v} \\ z \end{pmatrix}$ ,  $z(t, \theta) = P_2 \dot{u}(t + \theta)$ ,  $\theta \in (-\tau, 0)$  and

$$\mathcal{A}_d \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ z \end{pmatrix} = \begin{pmatrix} u_2 \\ v_2 \\ -A_1 u_1 - C(C^* u_1 + A_2^{\frac{1}{2}} v_1) - \alpha_1 BB^* u_2 - \alpha_2 BB^* z(-\tau) \\ -A_2^{\frac{1}{2}}(C^* u_1 + A_2^{\frac{1}{2}} v_1) \\ \partial_\theta z \end{pmatrix}, \quad (4.57)$$

with

$$\mathcal{D}(\mathcal{A}_d) := \left\{ (u_1, v_1, u_2, v_2, z) \in H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}} \times H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}} \times H^1(-\tau, 0; E_{1,\frac{1}{2}}), \right.$$

$$\left. A_1 u_1 + \alpha_1 BB^* u_2 + \alpha_2 BB^* z(-\tau) \in H_1, C^* u_1 + A_2^{\frac{1}{2}} v_1 \in H_{2,\frac{1}{2}}, z(0) = P_2 u_2 \right\}.$$

Assuming  $\alpha_2 \leq \alpha_1$ , we introduce in  $\tilde{\mathcal{H}}$  the new inner product

$$\left\langle \begin{pmatrix} U_1 \\ U_2 \\ z_1 \end{pmatrix}, \begin{pmatrix} V_1 \\ V_2 \\ z_2 \end{pmatrix} \right\rangle = \langle U_1, V_1 \rangle_{H_{\frac{1}{2}}} + \langle U_2, V_2 \rangle_H + \frac{\xi}{\tau} \int_{-\tau}^0 \langle B^* z_1(\theta), B^* z_2(\theta) \rangle_U d\theta,$$

where  $\xi$  is a constant satisfying

$$\tau\alpha_2 \leq \xi \leq \tau(2\alpha_1 - \alpha_2). \quad (4.58)$$

It can be seen easily that  $\tilde{\mathcal{H}}$  endowed with this inner product is a Hilbert space, and its associated norm is equivalent to the canonical norm of  $\tilde{\mathcal{H}}$ . Now, we are in the position to use the results in [1] to (4.48)-(4.50), and deduce first its well-posedness. To characterize the stabilization, we introduce the following delay energy functions

$$E_d(t) :=$$

$$\frac{1}{2} \|(w_1(t), w_2(t))\|_{H_{1,\frac{1}{2}} \times H_{2,\frac{1}{2}}}^2 + \frac{1}{2} \|(\dot{w}_1(t), \dot{w}_2(t))\|_{H_1 \times H_2}^2 + \frac{\xi}{2} \int_{-\tau}^0 \|B^* \dot{w}_1(t+\theta)\|_U^2 d\theta, \quad t \geq 0,$$

and

$$\tilde{E}_d(t) := \frac{1}{2} \|(u(t), v(t))\|_{H_{\frac{1}{2}}}^2 + \frac{1}{2} \|(\dot{u}(t), \dot{v}(t))\|_H^2 + \frac{\xi}{2} \int_{-\tau}^0 \|B^* \dot{u}(t+\theta)\|_U^2 d\theta, \quad t \geq 0.$$

Under the assumption (2.15),  $E_d(t)$  and  $\tilde{E}_d(t)$  are equivalent.

By our result [1, Theorem 1.1], Theorem 3.10 yields the following main result.

**Theorem 4.1.** *Assume that (2.9), (2.10), (2.11), (2.15) and (3.40) hold and that  $\alpha_2 < \alpha_1$ .*

*Then the following assertions are equivalent.*

1. *There are constants  $\omega, C > 0$  such that the system (4.48)-(4.50) satisfies the exponential decay*

$$E_d(t) \leq C e^{-\omega t} E_d(0) \text{ for all } (w_1^0, w_2^0, w_1^1, w_2^1, f_0) \in \mathcal{D}(\mathcal{A}_d).$$

2. *There exist  $T, c > 0$  such that*

$$c \int_0^T \|(B^* \phi)'\|_U^2 dt \geq \|A_1^{\frac{1}{2}} \phi(0)\|^2 + \|A_2^{\frac{1}{2}} \psi(0)\|^2 + \left\| \begin{pmatrix} \dot{\phi}(0) \\ \dot{\psi}(0) \end{pmatrix} \right\|_{H_1 \times H_2}^2$$

*for every solution  $(\phi, \psi)$  of the conservative adjoint system*

$$\ddot{\phi} + A_1 \phi + C \dot{\psi} = 0,$$

$$\ddot{\psi} + A_2 \psi - C^* \dot{\phi} = 0.$$

## 5 Applications

### 5.1 First example : Dirichlet boundary conditions

Consider the following coupled wave equations

$$\ddot{w}_1(t, x) - \frac{\partial^2 w_1}{\partial x^2}(t, x) + \alpha_1 \dot{w}_1(t, \xi) \delta_\xi + \alpha_2 \dot{w}_1(t - \tau, \xi) \delta_\xi + \beta \frac{\partial \dot{w}_2}{\partial x}(t, x) = 0, \quad (t, x) \in (0, \infty) \times ]0, 1[, \quad (5.59)$$

$$\ddot{w}_2(t, x) - \frac{\partial^2 w_2}{\partial x^2}(t, x) + \beta \frac{\partial \dot{w}_1}{\partial x}(t, x) = 0, \quad (t, x) \in (0, \infty) \times ]0, 1[, \quad (5.60)$$

$$w_i(t, 0) = w_i(t, 1) = 0, \quad t \in (0, \infty), \quad i = 1, 2, \quad (5.61)$$

$$w_i(0, x) = w_i^0(x), \quad \dot{w}_i(0, x) = w_i^1(x), \quad \dot{w}_1(s, x) = f_0(s, x), \quad -\tau \leq s < 0, \quad x \in ]0, 1[, \quad i = 1, 2, \quad (5.62)$$

with  $\xi \in (0, 1)$ ,  $\beta > 0$  and  $0 < \alpha_2 < \alpha_1$ . To put this control system into the framework of this paper, consider the spaces  $H_1 = H_2 = L^2(0, 1)$  and the operators  $A_1 = A_2 = -\frac{d^2}{dx^2}$ , with the domain  $\mathcal{D}(A_1) = \mathcal{D}(A_2) = H^2(0, 1) \cap H_0^1(0, 1)$  which are obviously self-adjoint positive operators. In this case, the domains of the fractional power operators are given by

$$\mathcal{D}(A_1^{\frac{1}{2}}) = \mathcal{D}(A_2^{\frac{1}{2}}) = H_0^1(0, 1).$$

The operator  $B$  and its adjoint  $B^*$  are given by

$$Bk = k\delta_\xi, \quad k \in \mathbb{R}, \quad B^*\varphi = \varphi(\xi), \quad \varphi \in H_0^1(0, 1)$$

and finally

$$C = \beta \frac{d}{dx}, \quad \text{with } \mathcal{D}(C) = H_0^1(0, 1).$$

It is clear that  $B^* : H_0^1(0, 1) \rightarrow \mathbb{R}$  is bounded and  $C^* = -\beta \frac{d}{dx}$  with

$$H_0^1(0, 1) \hookrightarrow \mathcal{D}(C^*) = H^1(0, 1). \quad (5.63)$$

Now assume that  $\beta < 1$ , then, with a simple integration by parts, the condition (2.15) is satisfied with constant  $\delta = \frac{\beta}{2}$ . Let us now check the assumption (3.40). Since in this example  $A_1 = A_2$ , we can easily see that

$$[\lambda^2 I + A_1 + \lambda^2 C(\lambda^2 + A_2)^{-1} C^*]^{-1} = \frac{1}{2} [\lambda^2 I + A_1 + \lambda C]^{-1} + \frac{1}{2} [\lambda^2 I + A_1 + \lambda C^*]^{-1}.$$

Thus, we have the following decomposition of the transfer function

$$H(\lambda) = \frac{\lambda}{2} B^* [\lambda^2 I + A_1 + \lambda C]^{-1} B + \frac{\lambda}{2} B^* [\lambda^2 I + A_1 + \lambda C^*]^{-1} B := H_1(\lambda) + H_2(\lambda).$$

For every  $k \in \mathbb{R}$ , the function

$$\psi := [\lambda^2 I + A_1 + \lambda C]^{-1} Bk$$

satisfies

$$\lambda^2 \psi(x) - \frac{d^2 \psi}{dx^2}(x) + \lambda \beta \frac{d \psi}{dx}(x) = 0, \quad x \in (0, \xi) \cup (\xi, 1) \quad (5.64)$$

$$\psi(0) = \psi(1) = 0, \quad (5.65)$$

$$[\psi]_\xi = 0, \left[ \frac{d\psi}{dx} \right]_\xi = k, \quad (5.66)$$

where we denote by  $[g]_\xi$  the jump of the function  $g$  at the point  $\xi$ . The solutions  $r_1, r_2$  of the equation  $r^2 - \beta\lambda r - \lambda^2 = 0$  are  $\frac{\lambda}{2}(\beta \pm \sqrt{\beta^2 + 4})$ . Hence, the solution of (5.64)-(5.65) is

$$\psi(x) = \begin{cases} A(e^{r_1 x} - e^{r_2 x}), & x \in (0, \xi) \\ B(e^{r_1(x-1)} - e^{r_2(x-1)}), & x \in (\xi, 1), \end{cases}$$

and (5.66) yields

$$\psi(x) = \frac{ke^{-\lambda\beta\xi}}{\lambda\sqrt{\beta^2 + 4}} \begin{cases} \frac{e^{r_1(\xi-1)} - e^{r_2(\xi-1)}}{e^{-r_2} - e^{-r_1}} (e^{r_1 x} - e^{r_2 x}), & x \in (0, \xi) \\ \frac{e^{r_1\xi} - e^{r_2\xi}}{e^{-r_2} - e^{-r_1}} (e^{r_1(x-1)} - e^{r_2(x-1)}), & x \in (\xi, 1). \end{cases}$$

Consequently

$$H_1(\lambda) = \frac{e^{-\lambda\beta\xi}}{2\sqrt{\beta^2 + 4}} \frac{e^{r_1(\xi-1)} - e^{r_2(\xi-1)}}{e^{-r_2} - e^{-r_1}} (e^{r_1\xi} - e^{r_2\xi})$$

and then, for every  $\gamma > 0$ , we have

$$\sup_{Re\lambda=2\gamma} |H_1(\lambda)| \leq \frac{1}{\sqrt{\beta^2 + 4}} \frac{\cosh(\gamma\sqrt{\beta^2 + 4}(1 - \xi))}{\sinh(\gamma\sqrt{\beta^2 + 4})} \cosh(\gamma\sqrt{\beta^2 + 4}\xi).$$

By similar calculus, we have the boundedness of  $H_2$ , and thus the assumption (3.40) is satisfied

Now, consider the conservative adjoint system

$$\frac{\partial^2 \phi}{\partial t^2}(t, x) - \frac{\partial^2 \phi}{\partial x^2}(t, x) + \beta \frac{\partial^2 \psi}{\partial x \partial t}(t, x) = 0, \quad (t, x) \in (0, \infty) \times ]0, 1[, \quad (5.67)$$

$$\frac{\partial^2 \psi}{\partial t^2}(t, x) - \frac{\partial^2 \psi}{\partial x^2}(t, x) + \beta \frac{\partial^2 \phi}{\partial x \partial t}(t, x) = 0, \quad (t, x) \in (0, \infty) \times ]0, 1[, \quad (5.68)$$

$$\phi(t, 0) = \psi(t, 0) = \phi(t, 1) = \psi(t, 1) = 0, \quad t \in (0, \infty), \quad (5.69)$$

$$\phi(0, x) = \phi^0(x), \quad \dot{\phi}(0, x) = \phi^1(x), \quad \psi(0, x) = \psi^0(x), \quad \dot{\psi}(0, x) = \psi^1(x), \quad x \in ]0, 1[. \quad (5.70)$$

Consider the initial conditions as follows

$$\phi^0(x) = \sum_{n \in \mathbb{Z}^*} a_n \cos \left( \frac{n\beta\pi}{\sqrt{\beta^2 + 4}} x \right) \sin(n\pi x), \quad \phi^1(x) = \sum_{n \in \mathbb{Z}^*} \lambda_n a_n \cos \left( \frac{n\beta\pi}{\sqrt{\beta^2 + 4}} x \right) \sin(n\pi x)$$

$$\psi^0(x) = \sum_{n \in \mathbb{Z}^*} a_n \sin \left( \frac{n\beta\pi}{\sqrt{\beta^2 + 4}} x \right) \sin(n\pi x), \quad \psi^1(x) = \sum_{n \in \mathbb{Z}^*} \lambda_n a_n \sin \left( \frac{n\beta\pi}{\sqrt{\beta^2 + 4}} x \right) \sin(n\pi x)$$

with  $(\lambda_n a_n)$  are in  $l^2(\mathbb{C})$ , where  $\lambda_n = \frac{2in\pi}{\sqrt{\beta^2 + 4}}$ ,  $\forall n \in \mathbb{Z}^*$ .

By standard technics, we obtain

$$\phi(t, x) = \sum_{n \in \mathbb{Z}^*} a_n e^{\lambda_n t} \cos \left( \frac{n\beta\pi}{\sqrt{\beta^2 + 4}} x \right) \sin(n\pi x)$$

and then,

$$\frac{\partial \phi}{\partial t}(t, \xi) = \sum_{n \in \mathbb{Z}^*} \lambda_n a_n e^{\lambda_n t} \cos\left(\frac{n\beta\pi}{\sqrt{\beta^2 + 4}}\xi\right) \sin(n\pi\xi).$$

Now, by the Ingham's inequality, for any  $T > \sqrt{\beta^2 + 4}$  we have

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(t, \xi) \right|^2 dt \asymp \sum_{n \in \mathbb{Z}^*} |\lambda_n|^2 |a_n|^2 \left| \cos\left(\frac{n\beta\pi}{\sqrt{\beta^2 + 4}}\xi\right) \right|^2 |\sin(n\pi\xi)|^2, \quad (5.71)$$

which implies (see [4] and [16] for more details), as in [5] for the only one string equation, that the system (5.59)-(5.62) is not exponentially stable in the energy space for all  $\xi$  and  $\beta$ .

## 5.2 Second example : mixed boundary conditions

Consider the following coupled wave equations

$$\ddot{w}_1(t, x) - \frac{\partial^2 w_1}{\partial x^2}(t, x) + w_1(t, x) + \alpha_1 \dot{w}_1(t, \xi) \delta_\xi + \alpha_2 \dot{w}_1(t - \tau, \xi) \delta_\xi + \beta \frac{\partial \dot{w}_2}{\partial x}(t, x) = 0, \quad (t, x) \in (0, \infty) \times (0, 1), \quad (5.72)$$

$$\ddot{w}_2(t, x) - \frac{\partial^2 w_2}{\partial x^2}(t, x) + w_2(t, x) + \beta \frac{\partial \dot{w}_1}{\partial x}(t, x) = 0, \quad (t, x) \in (0, \infty) \times (0, 1), \quad (5.73)$$

$$\frac{\partial w_1}{\partial x}(t, 0) = \frac{\partial w_1}{\partial x}(t, 1) = 0, \quad w_2(t, 0) = w_2(t, 1) = 0, \quad t \in (0, \infty), \quad (5.74)$$

$$w_i(0, x) = w_i^0(x), \quad \dot{w}_i(0, x) = w_i^1(x), \quad \dot{w}_1(s, x) = f_0(s, x), \quad -\tau \leq s < 0, \quad x \in (0, 1), \quad i = 1, 2. \quad (5.75)$$

with  $\xi \in (0, 1)$ ,  $\beta$  is a positive constant and  $0 < \alpha_2 < \alpha_1$ .

To put this control system into the framework of this paper, consider the spaces  $H_1 = H_2 = L^2(0, 1)$  and the operators  $A_1 = A_2 = -\frac{d^2}{dx^2} + I$ , with domains

$$\mathcal{D}(A_1) = \left\{ u \in H^2(0, 1), \frac{du}{dx}(0) = 0, \frac{du}{dx}(1) = 0 \right\}, \quad \mathcal{D}(A_2) = H^2(0, 1) \cap H_0^1(0, 1)$$

which are obviously self-adjoint positive operators. In this case, the domain of the fractional power operators are given by

$$\mathcal{D}(A_1^{\frac{1}{2}}) = H^1(0, 1), \quad \mathcal{D}(A_2^{\frac{1}{2}}) = H_0^1(0, 1).$$

The operator  $B$  and its adjoint  $B^*$  are given by

$$Bk = k\delta_\xi, \quad k \in \mathbb{R}, \quad B^*\varphi = \varphi(\xi), \quad \varphi \in H^1(0, 1)$$

and finally

$$C = \beta \frac{d}{dx}, \quad \text{with } \mathcal{D}(C) = H_0^1(0, 1).$$

It is clear that  $B^* : H^1(0, 1) \rightarrow \mathbb{R}$  is bounded and  $C^* = -\beta \frac{d}{dx}$  with

$$H_0^1(0, 1) \hookrightarrow \mathcal{D}(C^*) = H^1(0, 1). \quad (5.76)$$

Assuming  $\beta < 1$ , as in the first example, the condition (2.15) is satisfied with constant  $\delta = \frac{\beta}{2}$ . Let us verify the boundedness of the transfer function of the above system. For this, let  $k \in \mathbb{R}$ ,  $\operatorname{Re} \lambda > 0$  and the elliptic system

$$\lambda^2 \phi_1(x) - \frac{d^2 \phi_1}{dx^2}(x) + \phi_1(x) + \lambda \beta \frac{d\phi_2}{dx}(x) = k \delta_\xi, \quad x \in (0, 1), \quad (5.77)$$

$$\lambda^2 \phi_2(x) - \frac{d^2 \phi_2}{dx^2}(x) + \phi_2(x) + \lambda \beta \frac{d\phi_1}{dx}(x) = 0, \quad x \in (0, 1), \quad (5.78)$$

$$\frac{d\phi_1}{dx}(0) = \frac{d\phi_1}{dx}(1) = 0, \quad (5.79)$$

$$\phi_2(0) = \phi_2(1) = 0. \quad (5.80)$$

Then,

$$H(\lambda) = \lambda \phi_1(\xi) = \frac{\lambda}{2} \psi_1(\xi) + \frac{\lambda}{2} \psi_2(\xi) := H_1(\lambda) + H_2(\lambda),$$

where  $\psi_1 = \phi_1 - \phi_2$ ,  $\psi_2 = \phi_1 + \phi_2$  satisfy the following equations

$$\lambda^2 \psi_1(x) - \frac{d^2 \psi_1}{dx^2}(x) + \psi_1 + \lambda \beta \frac{d\psi_1}{dx}(x) = 0, \quad x \in (0, \xi) \cup (\xi, 1) \quad (5.81)$$

$$\lambda^2 \psi_2(x) - \frac{d^2 \psi_2}{dx^2}(x) + \psi_2 - \lambda \beta \frac{d\psi_2}{dx}(x) = 0, \quad x \in (0, \xi) \cup (\xi, 1) \quad (5.82)$$

$$\frac{d(\psi_1 + \psi_2)}{dx}(0) = \frac{d(\psi_1 + \psi_2)}{dx}(1) = 0, \quad (5.83)$$

$$(\psi_1 - \psi_2)(0) = (\psi_1 - \psi_2)(1) = 0, \quad (5.84)$$

$$[\psi_i]_\xi = 0, \quad \left[ \frac{d\psi_i}{dx} \right]_\xi = k, \quad i = 1, 2. \quad (5.85)$$

Let  $r_1, r_2$  be the roots of the equation  $r^2 - \beta \lambda r - \lambda^2 - 1 = 0$ , which are  $\frac{\beta \lambda}{2} \pm \sqrt{\frac{\beta^2 \lambda^2}{4} + \lambda^2 + 1}$ .

Then the solution of the equations (5.81)-(5.84) is given by

$$\psi_1(x) = \begin{cases} A_1 e^{r_1 x} + B_1 e^{r_2 x}, & x \in (0, \xi) \\ C_1 e^{r_1(x-1)} + D_1 e^{r_2(x-1)}, & x \in (\xi, 1) \end{cases}$$

and

$$\psi_2(x) = \begin{cases} A_1 e^{-r_1 x} + B_1 e^{-r_2 x}, & x \in (0, \xi) \\ C_1 e^{-r_1(x-1)} + D_1 e^{-r_2(x-1)}, & x \in (\xi, 1). \end{cases}$$

Therefore, (5.85) yields

$$\psi_1(x) = \frac{k}{r_1 - r_2} \begin{cases} \frac{e^{-2r_1(\xi-1)}+1}{e^{-r_1\xi}-e^{r_1(-\xi+2)}} e^{r_1 x} - \frac{e^{-2r_2(\xi-1)}+1}{e^{-r_2\xi}-e^{r_2(-\xi+2)}} e^{r_2 x}, & x \in (0, \xi), \\ \left( e^{r_1} \frac{e^{-2r_1(\xi-1)}+1}{e^{-r_1\xi}-e^{r_1(-\xi+2)}} + e^{-r_1(\xi-1)} \right) e^{r_1(x-1)} \\ - \left( e^{r_2} \frac{e^{-2r_2(\xi-1)}+1}{e^{-r_2\xi}-e^{r_2(-\xi+2)}} + e^{-r_2(\xi-1)} \right) e^{r_2(x-1)}, & x \in (\xi, 1) \end{cases}$$

and

$$\psi_2(x) = \frac{k}{r_1 - r_2} \begin{cases} \frac{e^{-2r_1(\xi-1)}+1}{e^{-r_1\xi}-e^{r_1(-\xi+2)}} e^{-r_1 x} - \frac{e^{-2r_2(\xi-1)}+1}{e^{-r_2\xi}-e^{r_2(-\xi+2)}} e^{-r_2 x}, & x \in (0, \xi), \\ \left( e^{r_1} \frac{e^{-2r_1(\xi-1)}+1}{e^{-r_1\xi}-e^{r_1(-\xi+2)}} + e^{-r_1(\xi-1)} \right) e^{-r_1(x-1)} \\ - \left( e^{r_2} \frac{e^{-2r_2(\xi-1)}+1}{e^{-r_2\xi}-e^{r_2(-\xi+2)}} + e^{-r_2(\xi-1)} \right) e^{-r_2(x-1)}, & x \in (\xi, 1). \end{cases}$$

Consequently

$$H_1(\lambda) = \frac{1}{2\sqrt{\beta^2 + 4}} \frac{-\cosh[r_1(\xi - 1)] \sinh(r_2) e^{r_1\xi} + \cosh[r_2(\xi - 1)] \sinh(r_1) e^{r_2\xi}}{\sinh(r_1) \sinh(r_2)},$$

$$H_2(\lambda) = \frac{1}{2\sqrt{\beta^2 + 4}} \frac{-\cosh[r_1(\xi - 1)] \sinh(r_2) e^{-r_1\xi} + \cosh[r_2(\xi - 1)] \sinh(r_1) e^{-r_2\xi}}{\sinh(r_1) \sinh(r_2)}.$$

As  $r_1$  and  $r_2$  behave asymptotically as  $r_3 := \frac{\beta\lambda}{2} + \frac{\lambda}{2}\sqrt{\beta^2 + 4}$  and  $r_4 := \frac{\beta\lambda}{2} - \frac{\lambda}{2}\sqrt{\beta^2 + 4}$ , respectively it suffieses to see that for  $r_3, r_4$ , one has

$$\sup_{Re\lambda=2\gamma} |H_1(\lambda)| \leq \frac{1}{\sqrt{\beta^2 + 4}} \frac{\cosh[\gamma(\xi - 1)(\beta + \sqrt{\beta^2 + 4})] \cosh[\gamma(\beta + \sqrt{\beta^2 + 4})] e^{\gamma\xi(\beta + \sqrt{\beta^2 + 4})}}{\sinh[\gamma(\beta + \sqrt{\beta^2 + 4})] \sinh[\gamma(-\beta + \sqrt{\beta^2 + 4})]}.$$

By similar calculus, we have the boundedness of  $H_2$ , and this achieves the claim.

Consider the conservative adjoint system

$$\begin{aligned} \frac{\partial^2 \phi}{\partial t^2}(t, x) - \frac{\partial^2 \phi}{\partial x^2}(t, x) + \phi(t, x) + \beta \frac{\partial^2 \psi}{\partial x \partial t}(t, x) &= 0, \quad (t, x) \in (0, \infty) \times (0, 1), \\ \frac{\partial^2 \psi}{\partial t^2}(t, x) - \frac{\partial^2 \psi}{\partial x^2}(t, x) + \psi(t, x) + \beta \frac{\partial^2 \phi}{\partial x \partial t}(t, x) &= 0, \quad (t, x) \in (0, \infty) \times (0, 1), \\ \frac{\partial \phi}{\partial x}(t, 0) = \psi(t, 0) &= \frac{\partial \phi}{\partial x}(t, 1) = \psi(t, 1) = 0, \quad t \in (0, \infty), \\ \phi(0, x) = \phi^0(x), \quad \dot{\phi}(0, x) = \phi^1(x), \quad \psi(0, x) = \psi^0(x), \quad \dot{\psi}(0, x) = \psi^1(x), &\quad x \in (0, 1). \end{aligned}$$

The initial conditions can be written as

$$\begin{aligned} \phi^0(x) &= \sum_{n \in \mathbb{Z}^*} a_n \cos(n\pi x), \quad \phi^1(x) = \sum_{n \in \mathbb{Z}^*} \lambda_n a_n \cos(n\pi x) \\ \psi^0(x) &= \sum_{n \in \mathbb{Z}^*} a_n \sin(n\pi x), \quad \psi^1(x) = \sum_{n \in \mathbb{Z}^*} \lambda_n a_n \sin(n\pi x) \end{aligned}$$

with  $\lambda_n = i \frac{2n\pi\beta}{\beta^2 + 4} \pm i \sqrt{\frac{4n^2\pi^2\beta^2}{(\beta^2 + 4)^2} + \frac{4+4n^2\pi^2}{\beta^2 + 4}}$ ,  $n \in \mathbb{Z}^*$ , and  $(\lambda_n a_n) \in l^2(\mathbb{C})$ . Hence, by standard technics, we obtain

$$\phi(t, x) = \sum_{n \in \mathbb{Z}^*} a_n e^{\lambda_n t} \cos(n\pi x),$$

and then

$$\frac{\partial \phi}{\partial t}(t, \xi) = \sum_{n \in \mathbb{Z}^*} \lambda_n a_n e^{\lambda_n t} \cos(n\pi \xi).$$

Now, by the Ingham's inequality, for any  $T > \frac{\beta^2 + 4}{\beta + \sqrt{\beta^2 + 4}}$  there is  $C_{T, \xi, \beta} > 0$  such that

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(t, \xi) \right|^2 dt \geq C_{T, \xi, \beta} \sum_{n \in \mathbb{Z}^*} |\lambda_n|^2 |a_n|^2 |\cos(n\pi \xi)|^2.$$

Finally, this implies, as in [1, 5] for the only one string equation, that the system is exponentially stable in the energy space if and only if  $\xi$  is a rational number with coprime factorisation  $\xi = \frac{p}{q}$ , where  $p$  is odd.

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